MBF1243 | Derivatives
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L1 – Determination of Forward and Futures Prices

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Consumption vs Investment Assets

- When considering forward and futures contracts, it is important to distinguish between investment assets and consumption assets.

- **Investment asset** - asset that is held for investment purposes by significant numbers of investors. (Eg, stocks and bonds are clearly investment assets. Gold and silver are also investment assets.

  - Note that investment assets do not have to be held exclusively for investment. (Silver, for example, has a number of industrial uses.) However, they do have to satisfy the requirement that they are held by significant numbers of investors solely for investment.

- **Consumption asset** is an asset that is held primarily for consumption. It is not usually held for investment. Examples of consumption assets are commodities such as copper, oil, and pork bellies.
Short Selling

- Short selling involves selling securities you do not own. Also referred to as “shorting”

- Eg. An investor instructs a broker to short 500 IBM shares. The broker will carry out the instructions by borrowing the shares from another client and selling them in the market in the usual way.

- The investor can maintain the short position for as long as desired, provided there are always shares for the broker to borrow.

- At some stage, however, the investor will close out the position by purchasing 500 IBM shares. These are then replaced in the account of the client from which the shares were borrowed.
Short Selling

• The investor takes a profit if the stock price has declined and a loss if it has risen.

• If at any time while the contract is open the broker is not able to borrow shares, the investor is forced to close out the position, even if not ready to do so.

• Sometimes a fee is charged for lending shares or other securities to the party doing the shorting.

• An investor with a short position must pay to the broker any income, such as dividends or interest, that would normally be received on the securities that have been shorted.

• The broker will transfer this income to the account of the client from whom the securities have been borrowed.
Example

- Consider the position of an investor who shorts 500 shares in April when the price per share is $120 and closes out the position by buying them back in July when the price per share is $100.

- Suppose that a dividend of $1 per share is paid in May.

- The investor receives $500 \times $120 = $60,000 in April when the short position is initiated.

- The dividend leads to a payment by the investor of $500 \times $1 = $500 in May.

- The investor also pays $500 \times $100 = $50,000 for shares when the position is closed out in July.

- The net gain, therefore, is $60,000 - $500 - $50,000 = $9,500
## Example

### Table 5.1  Cash flows from short sale and purchase of shares.

#### Purchase of shares

<table>
<thead>
<tr>
<th>Month</th>
<th>Description</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>April</td>
<td>Purchase 500 shares for $120</td>
<td>-$60,000</td>
</tr>
<tr>
<td>May</td>
<td>Receive dividend</td>
<td>+$500</td>
</tr>
<tr>
<td>July</td>
<td>Sell 500 shares for $100 per share</td>
<td>+$50,000</td>
</tr>
<tr>
<td></td>
<td><strong>Net profit</strong></td>
<td>-$9,500</td>
</tr>
</tbody>
</table>

#### Short sale of shares

<table>
<thead>
<tr>
<th>Month</th>
<th>Description</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>April</td>
<td>Borrow 500 shares and sell them for $120</td>
<td>+$60,000</td>
</tr>
<tr>
<td>May</td>
<td>Pay dividend</td>
<td>-$500</td>
</tr>
<tr>
<td>July</td>
<td>Buy 500 shares for $100 per share</td>
<td>-$50,000</td>
</tr>
<tr>
<td></td>
<td>Replace borrowed shares to close short position</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Net profit</strong></td>
<td>+$9,500</td>
</tr>
</tbody>
</table>
Notation for Valuing Futures and Forward Contracts

\( S_0 \): Spot price today

\( F_0 \): Price of the asset underlying the forward or futures contract today

\( T \): Time until delivery date in a forward or futures contract (in years)

\( r \): Risk-free interest rate rate for maturity T
An Arbitrage Opportunity?

- Consider a long forward contract to purchase a non-dividend-paying stock in 3 months

- The spot price of a non-dividend-paying stock is $40
  - The 3-month forward price is $43
  - The 3-month US$ interest rate is 5% per annum

- Is there an arbitrage opportunity?
  - An arbitrageur can borrow $40 at the risk-free interest rate of 5% per annum
  - Buy one share, and short a forward contract to sell one share in 3 months.
  - At the end of the 3 months, the arbitrageur delivers the share and receives $43.
  - The sum of money required to pay off the loan is
    \[ 40e^{0.05 \times 3/12} = 40.50 \]
  - By following this strategy, the arbitrageur locks in a profit of
    - $43.00 - $40.50 = $2.50 at the end of the 3-month period.
Another Arbitrage Opportunity?

• Suppose that:
  ➢ The spot price of nondividend-paying stock is $40
  ➢ The 3-month forward price is $39
  ➢ The 1-year $ interest rate is 5% per annum (continuously compounded)

• Is there an arbitrage opportunity?

➢ An arbitrageur can short one share,
➢ Invest the proceeds of the short sale at 5% per annum for 3 months,
➢ and take a long position in a 3-month forward contract.
➢ The proceeds of the short sale grow to \(40e^{0.05 \times 3/12}\), or $40.50 in 3 months.
➢ At the end of the 3 months, the arbitrageur pays $39, takes delivery of the share under the terms of the forward contract, and uses it to close out the short position.
➢ A net gain of \$40.50 - \$39.00 = \$1.50\ is therefore made at the end of the 3 months.
Another Arbitrage Opportunity?

- The two trading strategies we have considered are summarized in Table 5.2.

- Under what circumstances do arbitrage opportunities such as those in Table 5.2 not exist?

- The first arbitrage works when the forward price is greater than $40.50.

- The second arbitrage works when the forward price is less than $40.50. We deduce that for there to be no arbitrage the forward price must be exactly $40.50.
Another Arbitrage Opportunity?

Table 5.2  Arbitrage opportunities when forward price is out of line with spot price for asset providing no income. (Asset price = $40; interest rate = 5%; maturity of forward contract = 3 months.)

<table>
<thead>
<tr>
<th>Forward Price = $43</th>
<th>Forward Price = $39</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action now:</strong></td>
<td><strong>Action now:</strong></td>
</tr>
<tr>
<td>Borrow $40 at 5% for 3 months</td>
<td>Short 1 unit of asset to realize $40</td>
</tr>
<tr>
<td>Buy one unit of asset</td>
<td>Invest $40 at 5% for 3 months</td>
</tr>
<tr>
<td>Enter into forward contract to sell asset in 3 months for $43</td>
<td>Enter into a forward contract to buy asset in 3 months for $39</td>
</tr>
<tr>
<td><strong>Action in 3 months:</strong></td>
<td><strong>Action in 3 months:</strong></td>
</tr>
<tr>
<td>Sell asset for $43</td>
<td>Buy asset for $39</td>
</tr>
<tr>
<td>Use $40.50 to repay loan with interest</td>
<td>Close short position</td>
</tr>
<tr>
<td></td>
<td>Receive $40.50 from investment</td>
</tr>
</tbody>
</table>

Profit realized = $2.50  
Profit realized = $1.50
A Generalization

If the spot price of an investment asset is $S_0$ and the futures price for a contract deliverable in $T$ years is $F_0$, then

$$F_0 = S_0 e^{rT} \quad (5.1)$$

where $r$ is the $T$-year risk-free rate of interest.

In our examples, $S_0 = 40$, $T = 0.25$, and $r = 0.05$ so that

$$F_0 = 40 e^{0.05 \times 0.25} = 40.50$$
A Generalization

• If $F_0 > S_0e^{rT}$, arbitrageurs can buy the asset and short forward contracts on the asset.

• If $F_0 < S_0e^{rT}$, they can short the asset and enter into long forward contracts on it.
A Generalization

Formula still works for an investment asset because investors who hold the asset will sell it and buy forward contracts when the forward price is too low.
A Generalization

• Consider a 4-month forward contract to buy a zero-coupon bond that will mature 1 year from today. (This means that the bond will have 8 months to go when the forward contract matures.)

• The current price of the bond is $930.

• We assume that the 4-month risk-free rate of interest (continuously compounded) is 6% per annum.

• Because zero-coupon bonds provide no income, we can use equation (5.1)

  with \( T = 4/12 \), \( r = 0.06 \), and \( S_0 = 930 \). The forward price, \( F_0 \), is given by

  \[
  F_0 = 930e^{0.06 \times 4/12} = 948.79
  \]

• This would be the delivery price in a contract negotiated today.
When an Investment Asset Provides a Known Income

• Consider a forward contract on an investment asset that will provide a perfectly predictable cash income to the holder.

• Examples are stocks paying known dividends and coupon-bearing bonds.

• Consider a long forward contract to purchase a coupon-bearing bond whose current price is $900.
  ➢ Assume the forward contract matures in 9 months.
  ➢ Assume that a coupon payment of $40 is expected after 4 months.
  ➢ Assume that the 4-month and 9-month risk-free interest rates (continuously compounded) are, respectively, 3% and 4% per annum.
When an Investment Asset Provides a Known Income

• Suppose first that the forward price is relatively high at $910.

• An arbitrageur can borrow $900 to buy the bond and short a forward contract.

• The coupon payment has a present value of \(40e^{0.03 \times 4/12} = $39.60\).

• Of the $900, $39.60 is therefore borrowed at 3% per annum for 4 months so that it can be repaid with the coupon payment.

• The remaining $860.40 is borrowed at 4% per annum for 9 months.

• The amount owing at the end of the 9-month period is \(860.40e^{0.04 \times 0.75} = $886.60\).

• A sum of $910 is received for the bond under the terms of the forward contract. The arbitrageur therefore makes a net profit of \(910.0 - 886.60 = $23.40\).
When an Investment Asset Provides a Known Income

• Suppose next that the forward price is relatively low at $870.

• An investor can short the bond and enter into a long forward contract.

• Of the $900 realized from shorting the bond, $39.60 is invested for 4 months at 3% per annum so that it grows into an amount sufficient to pay the coupon on the bond.

• The remaining $860.40 is invested for 9 months at 4% per annum and grows to $886.60.

• Under the terms of the forward contract, $870 is paid to buy the bond and the short position is closed out.

• The investor therefore gains $886.60 - $870 = $16.60
When an Investment Asset Provides a Known Income

• The two strategies we have considered are summarized in Table 5.3.3
• The first strategy in Table 5.3 produces a profit when the forward price is greater than $886.60, whereas the second strategy produces a profit when the forward price is less than $886.60.
• It follows that if there are no arbitrage opportunities then the forward price must be $886.60.
When an Investment Asset Provides a Known Income

Table 5.3  Arbitrage opportunities when 9-month forward price is out of line with spot price for asset providing known cash income. (Asset price = $900; income of $40 occurs at 4 months; 4-month and 9-month rates are, respectively, 3% and 4% per annum.)

<table>
<thead>
<tr>
<th>Forward price = $910</th>
<th>Forward price = $870</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action now:</strong></td>
<td><strong>Action now:</strong></td>
</tr>
<tr>
<td>Borrow $900: $39.60 for 4 months and $860.40 for 9 months</td>
<td>Short 1 unit of asset to realize $900</td>
</tr>
<tr>
<td>Buy 1 unit of asset</td>
<td>Invest $39.60 for 4 months and $860.40 for 9 months</td>
</tr>
<tr>
<td>Enter into forward contract to sell asset in 9 months for $910</td>
<td>Enter into a forward contract to buy asset in 9 months for $870</td>
</tr>
<tr>
<td><strong>Action in 4 months:</strong></td>
<td><strong>Action in 4 months:</strong></td>
</tr>
<tr>
<td>Receive $40 of income on asset</td>
<td>Receive $40 from 4-month investment</td>
</tr>
<tr>
<td>Use $40 to repay first loan with interest</td>
<td>Pay income of $40 on asset</td>
</tr>
<tr>
<td><strong>Action in 9 months:</strong></td>
<td><strong>Action in 9 months:</strong></td>
</tr>
<tr>
<td>Sell asset for $910</td>
<td>Receive $886.60 from 9-month investment</td>
</tr>
<tr>
<td>Use $886.60 to repay second loan with interest</td>
<td>Buy asset for $870</td>
</tr>
<tr>
<td>Profit realized = $23.40</td>
<td>Profit realized = $16.60</td>
</tr>
</tbody>
</table>
When an Investment Asset Provides a Known Income

We can generalize from this example to argue that, when an investment asset will provide income with a present value of \( I \) during the life of a forward contract, we have

\[
F_0 = (S_0 - I) e^{rT} \tag{5.2}
\]

where \( I \) is the present value of the income during life of forward contract.

In our example,
\[S_0 = 900.00, \quad I = 40e^{-0.03 \times 4/12} = 39:60,\]

\[r = 0.04, \quad \text{and} \quad T = 0.75, \quad \text{so that} \]
\[F_0 = (900.00 - 39:60) e^{0.04 \times 0.75} = \$886:60\]
When an Investment Asset Provides a Known Income

Equation (5.2) applies to any investment asset that provides a known cash income.

If $F_0 > (S_0 - I)e^{rT}$, an arbitrageur can lock in a profit by buying the asset and shorting a forward contract on the asset;

if $F_0 < (S_0 - I)e^{rT}$, an arbitrageur can lock in a profit by shorting the asset and taking a long position in a forward contract.

If short sales are not possible, investors who own the asset will find it profitable to sell the asset and enter into long forward contracts.
When an Investment Asset Provides a Known Income

Consider a 10-month forward contract on a stock when the stock price is $50.

We assume that the risk-free rate of interest (continuously compounded) is 8% per annum for all maturities.

We also assume that dividends of $0.75 per share are expected after 3 months, 6 months, and 9 months.

The present value of the dividends, I, is

\[ I = 0.75e^{0.08 \times 3/12} + 0.75e^{0.08 \times 6/12} + 0.75e^{0.08 \times 9/12} = 2.162 \]

The variable T is 10 months, so that the forward price, \( F_0 \), from equation (5.2), is given by

\[ F_0 = (50 - 2.162)e^{0.08 \times 10/12} = 51.14 \]

If the forward price were less than this, an arbitrageur would short the stock and buy forward contracts. If the forward price were greater than this, an arbitrageur would short forward contracts and buy the stock in the spot market.
When an Investment Asset Provides a Known Yield

• Consider the situation where the asset underlying a forward contract provides a known yield rather than a known cash income.
  ➢ This means that the income is known when expressed as a percentage of the asset’s price at the time the income is paid.

• Suppose that an asset is expected to provide a yield of 5% per annum. This could mean that income is paid once a year and is equal to 5% of the asset price at the time it is paid, in which case the yield would be 5% with annual compounding.

  ➢ Alternatively, it could mean that income is paid twice a year and is equal to 2.5% of the asset price at the time it is paid, in which case the yield would be 5% per annum with semiannual compounding.
When an Investment Asset Provides a Known Yield

- In Section 4.2 we explained that we will normally measure interest rates with continuous compounding. Similarly, we will normally measure yields with continuous compounding.

- Define $q$ as the average yield per annum on an asset during the life of a forward contract with continuous compounding.

- It can be shown (see Problem 5.20) that

$$F_0 = S_0 e^{(r-q)T}$$  \hspace{1cm} 5.3
When an Investment Asset Provides a Known Yield

Example 5.3

Consider a 6-month forward contract on an asset that is expected to provide income equal to 2% of the asset price once during a 6-month period.

The risk free rate of interest (with continuous compounding) is 10% per annum.

The asset price is $25.

In this case, $S_0 = 25$, $r = 0.10$, and $T = 0.5$.

The yield is 4% per annum with semiannual compounding, this is 3.96% per annum with continuous compounding.

It follows that $q = 0.0396$, so that from equation (5.3) the forward price, $F_0$, is given by

$$F_0 = 25e^{(0.10 - 0.0396) \times 0.5} = 25.77$$
Valuing a Forward Contract

• The value of a forward contract at the time it is first entered into is zero.

• At a later stage, it may prove to have a positive or negative value. It is important for banks and other financial institutions to value the contract each day. (This is referred to as marking to market the contract.)

• Using the notation introduced earlier, we suppose
  ➢ $K$ is the delivery price for a contract that was negotiated some time ago,
  ➢ the delivery date is $T$ years from today, and
  ➢ $r$ is the $T$-year risk-free interest rate.

• The variable $F_0$ is the forward price that would be applicable if we negotiated the contract today. In addition, we define $f$ to be the value of forward contract today.
Valuing a Forward Contract

• By considering the difference between a contract with delivery price $K$ and a contract with delivery price $F_0$ we can deduce that:

  ➢ the value of a long forward contract is 
    $$(F_0 - K)e^{-rT}$$

  ➢ the value of a short forward contract is 
    $$(K - F_0)e^{-rT}$$
Valuing a Forward Contract

Example 5.4

• A long forward contract on a non-dividend-paying stock was entered into some time ago.
• It currently has 6 months to maturity.
• The risk-free rate of interest (with continuous compounding) is 10% per annum, the stock price is $25, and
• The delivery price is $24.

• In this case, $S_0 = 25$, $r = 0.10$, $T = 0.5$, and $K = 24$.

• From equation (5.1), the 6-month forward price, $F_0$, is given by
  \[ F_0 = 25e^{0.1 \times 0.5} = 26.28 \]

• From equation (5.4), the value of the forward contract is
  \[ f = (26.28 - 24)e^{0.1 \times 0.5} = 2.17 \]
Can be viewed as an investment asset paying a dividend yield

The futures price and spot price relationship is therefore

\[ F_0 = S_0 e^{(r-q)T} \]

where \( q \) is the average dividend yield on the portfolio represented by the index during life of contract.
### Futures Prices & Future Spot Prices (continued)

<table>
<thead>
<tr>
<th>Risk Type</th>
<th>Condition</th>
<th>Price Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Systematic Risk</td>
<td>$k = r$</td>
<td>$F_0 = E(S_T)$</td>
</tr>
<tr>
<td>Positive Systematic Risk</td>
<td>$k &gt; r$</td>
<td>$F_0 &lt; E(S_T)$</td>
</tr>
<tr>
<td>Negative Systematic Risk</td>
<td>$k &lt; r$</td>
<td>$F_0 &gt; E(S_T)$</td>
</tr>
</tbody>
</table>

Positive systematic risk: stock indices  
Negative systematic risk: gold (at least for some periods)